University of Groningen

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Takens, Floris; Verbitski, Evgeni

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MULTIFRACTAL ANALYSIS OF LOCAL ENTROPIES FOR GIBBS MEASURES

Floris Takens and Evgeni Verbitski (Groningen)

Abstract

Recently in the series of papers L. Barreira, Ya.B. Pesin, J. Schmeling and H. Weiss performed a complete multifractal analysis of local dimensions, entropies and Lyapunov exponents of conformal expanding maps and surface Axiom A diffeomorphisms for Gibbs measures. The main goal of these papers was primarily the analysis of the local (pointwise) dimensions. This is an extremely difficult problem and, for example, similar results for hyperbolic systems in dimensions 3 and higher have not been yet obtained.

In the present work we concentrate our attention on the multifractal analysis of the local (pointwise) entropies. We are able to obtain results, which are similar to those mentioned above, for Gibbs measures of the expansive homeomorphisms with specification property. Note that such homeomorphisms may not have Markov partitions, which is a crucial condition in the previous works. However, due to the fact that less is known about thermodynamical properties of these dynamical systems we were able to obtain only the continuous differentiabilty of the multifractal spectrum of local entropies (compare: the same spectra for the dynamical systems with Markov partitions are analytic). We believe that the smoothness of the multifractal spectrum in our case can be improved.

We have related the mutifractal spectrum of the local entropies to the the spectrum of correlation entropies. These correlation entropies serve as entopy-like analogs of the Hentshel–Procaccia and Rényi spectra of generalized dimensions. This allows us to complete the duality between the mutifractal analyses of local dimensions and entropies.

Complete proofs can be found in [TV98] and will appear elsewhere.

Key words and phrases. Multifractal analysis, local entropies, thermodynamical formalism.

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1. Local entropies

Consider a continuous map f on a compact metric space (X, d). Suppose that f preserves a Borel non-atomic probability measure μ. Following [BK83] we define the lower (upper) local entropies at point $x \in X$ as

$$\underline{h}_\mu(f,x) := \lim_{\varepsilon\to 0} \liminf_{n\to\infty} -\frac{1}{n}\log\mu(B_n(x,\varepsilon)),$$
$$\overline{h}_\mu(f,x) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n}\log\mu(B_n(x,\varepsilon)).$$

where

$$B_n(x,\varepsilon) = \{y \in X : \; d(f^i(x), f^i(y)) < \varepsilon, \; i = 0, \dots, n-1\}.$$

If

$$\underline{h}_\mu(f,x) = \overline{h}_\mu(f,x)$$

we denote the common value by $h_\mu(f,x)$ and say that the local entropy at x exists.

The basic properties of local entropies are given by the following theorem due to Brin & Katok [BK83].

THEOREM 1.1 (Brin-Katok formula). *Let f, μ be as above, then*

1. *for μ-a.e. $x \in X$ the local entropy exists, i.e.,*
$$h_\mu(f,x) = \underline{h}_\mu(f,x) = \overline{h}_\mu(f,x);$$

2. *$h_\mu(f,x)$ is a f-invariant function of x, and*
$$\int h_\mu(f,x)\, d\mu = h_\mu(f),$$
where $h_\mu(f)$ is the measure-theoretic entropy of f.

One can consider a decomposition of X by the level sets of local entopies

$$K_\alpha = \{x \in X : \; h_\mu(f,x) = \alpha\}, \quad \alpha \geq 0,$$
$$X = \bigcup_{\alpha\geq 0} K_\alpha \cup \{x : h_\mu(f,x) \text{ does not exist }\}.$$

We shall use the topological entropy to measure the 'size' of sets $\{K_\alpha\}$. The topological entropy for non-compact or non-invariant sets has been introduced by Bowen [Bow73] as a dynamical analog of the Hausdorff dimension. An equivalent definition and its basic properties can be also found in [Pes97].

We define a multifractal spectrum (spectrum of singularities) for local entropies as follows

$$\mathcal{E}_E(\alpha) = h_{top}(f|_{K_\alpha}), \tag{1.1}$$

where $h_{top}(f|_Z)$ is the the topological entropy of Z. This notation needs a brief explanation: two E's stand for the topological $\mathcal{E}$ntropy of the level set of local Entropies. Other multifractal spectra are denoted by $\mathcal{D}_E, \mathcal{E}_D, \mathcal{D}_D$ see [Bow75].

2. Dynamics and measures

Multifractal analysis of local entropies is concerned with dynamical systems invariant measures. In this section we introduce expansive homeomorphisms with the specification property and Gibbs measures.

DEFINITION 2.1. A homeomorphism $f : X \to X$ is called expansive if there exists a constant $\gamma > 0$ such that if

$$d(f^n(x), f^n(y)) < \gamma \quad \text{for all } n \in \mathbb{Z} \qquad \text{then} \qquad x = y.$$

The maximal γ with such property is called the expansivity constant.

Another important requirement is the following.

DEFINITION 2.2 [Bow75]. We say that a homeomorphism $f : X \to X$ has the specification property (abbreviated to 'a homeomorphism f with specification') if for each $\delta > 0$ there exists an integer

$$p = p(\delta)$$

such that the following holds:

If

a) $I_1, \dots, I_n$ are intervals of integers, $I_j \subseteq [a, b]$ for some $a, b \in \mathbb{Z}$ and all j,

b)

$$\operatorname{dist}(I_i, I_j) \geq p(\delta)$$

for

$$i \neq j,$$

then for arbitrary $x_1, \dots, x_n \in X$ there exists a point $x \in X$ such that

1) $f^{b-a+p(\delta)}(x) = x$,

2) $d(f^k(x), f^k(x_i)) < \delta$ for $k \in I_i$.

The specification property guaranties good mixing properties of f and the existence of a sufficient number of periodic orbits.

We consider Gibbs measures for special, but quite large, class of potentials $\mathcal{V}_f(X)$. Namely, we say that $\varphi \in \mathcal{V}_f(X)$ if it is continuous and there exist $\varepsilon > 0$ and $K > 0$ such that for all $n \in \mathbb{N}$

$$d(f^k(x), f^k(y)) < \varepsilon$$

$$\text{for } k = 0, \dots, n-1 \Rightarrow \left|\sum_{i=0}^{n-1} \varphi(f^i(x)) - \sum_{i=0}^{n-1} \varphi(f^i(y))\right| < K.$$

For example, for a hyperbolic diffeomorphism f any Hölder continuous function φ is in $\mathcal{V}_f(X)$ [KH95, Prop. 20.2.6].

The following theorem states the existence and basic properties of the equilibrium states for the potentials from $\mathcal{V}_f(X)$.

THEOREM 2.3 [Bow75, HR92, KH95]. *If f is an expansive homeomorphism with specification and $\varphi \in \mathcal{V}_f(X)$ then there exists a unique measure μ_φ, called the equilibrium state of φ, such that*

$$P(\varphi) = h_{\mu_\varphi}(f) + \int \varphi d\mu_\varphi,$$

where $P(\varphi)$ is the topological pressure of φ.

The equilibrium state μ_φ is a Gibbs measure as well, i.e. for sufficiently small $\varepsilon > 0$ there exist constants $A_\varepsilon, B_\varepsilon > 0$ such that for all $x \in X$ and $n \geq 0$

$$A_\varepsilon \leq \frac{\mu_\varphi(B_n(x,\varepsilon))}{\exp\left(-nP(\varphi) + \sum_{i=0}^{n-1} \varphi(f^i(x))\right)} \leq B_\varepsilon. \tag{2.1}$$

Moreover, μ_φ is ergodic, positive on open sets and mixing.

3. Pressure function

A multifractal formalism usually relates by means of Legendre transform the multifractal spectrum as a function of α to a certain pressure function. The only way to establish smoothness or convexity of the spectrum of singularities is to show that the pressure function has these properties. Therefore any multifractal analysis contains, as important ingredient, the analysis of the pressure functions. In our case the result is given by the following lemma.

LEMMA 3.1. *Suppose $f : X \to X$ is an expansive homeomorphism with specification and $\varphi \in \mathcal{V}_f(X)$. Then the function $P(q\varphi)$ is continuously differentiable with respect to q and its derivative is given by*

$$\frac{dP(q\varphi)}{dq} = \int \varphi d\mu_q,$$

where μ_q is the equlibrium state corresponding to the potential $q\varphi$. Moreover, $P(q\varphi)$ is a strictly convex function of q provided the equilibrium state μ_φ for φ is not a measure of maximal entropy.

In the case μ_φ is the measure of maximal entropy

$$P(q\varphi) - qP(\varphi) = (1-q)h_{top}(f)$$

for all $q \in \mathbb{R}$.

REMARK. Much stronger results on smoothness of the pressure are known. For example, the analyticity of pressure function has been established for Smale spaces [Rue78], i.e., generalizations of Axiom A diffeomorphisms. The key property, which these systems inherit from hyperbolic dynamical systems, is the so-called local product structure, which in its turn guarantees the existence of Markov partitions. The known methods of establishing the analyticity of the pressure strongly rely on this Markov structure. Expansive homeomorphisms with T specification do not necessarily have Markov partitions. We were able to prove only C^1 of the pressure function for the expansive homeomorpshisms. However, we believe that this resut can be improved.

4. Statement of the result

THEOREM 4.1. *Let f be an expansive homeomorphism with the specification property of a compact metric space (X, d) with finite topological entropy. Let*

$$\varphi \in \mathcal{V}_f(X) \quad \textit{and} \quad \mu = \mu_\varphi$$

be the corresponding equilibrium state. Then

1. *For μ-a.e. $x \in X$ the local entropy at x exists and*

$$h_\mu(f, x) = h_\mu(f) = P(\varphi) - \int \varphi \, d\mu.$$

2. *For any $q \in \mathbb{R}$ define function*

$$T(q) = P(q\varphi) - qP(\varphi).$$

Then $T(q)$ is a convex C^1 function of q. Moreover,

$$T(0) = h_{top}(f), \quad T(1) = 0,$$

for every $q \in \mathbb{R}$ one has

$$T'(q) = \int \varphi d\mu_q - P(\varphi) \leq 0,$$

where μ_q is the equilibrium state for

$$\varphi_q = q\varphi.$$

3. *Put*

$$\alpha(q) = -T'(q).$$

Then

$$\mathcal{E}_E(\alpha(q)) := h_{top}(f|_{K_{\alpha(q)}}) = T(q) + q\alpha(q).$$

Define

$$\underline{\alpha} = \inf_q \alpha(q) = \lim_{q \to +\infty} \alpha(q),$$
$$\overline{\alpha} = \sup_q \alpha(q) = \lim_{q \to -\infty} \alpha(q).$$

Then

$$K_\alpha = \emptyset$$

if

$$\alpha \notin [\underline{\alpha}, \overline{\alpha}].$$

It means that the domain of the multifractal spectrum for local entropies $\alpha \to \mathcal{E}_E(\alpha)$ is the range of the function $q \to -T'(q)$.

4. *If the equilibrium state μ for the potential φ is not a measure of maximal entropy then the relation between $\mathcal{E}_E$ and $T(q)$ can be written in the following variational form*

$$\mathcal{E}_E(\alpha) = \inf_{q\in\mathbb{R}} (T(q) + q\alpha) \quad \textit{for } \alpha \in (\underline{\alpha}, \overline{\alpha}),$$
$$T(q) = \sup_{\alpha\in(\underline{\alpha},\overline{\alpha})} (\mathcal{E}_E(\alpha) - q\alpha) \quad \textit{for } q \in \mathbb{R}.$$

This implies that $\mathcal{E}_E$ is strictly concave and continuously differentiable on $(\underline{\alpha}, \overline{\alpha})$ with derivative given by

$$\mathcal{E}'_E(\alpha) = q,$$

where $q \in \mathbb{R}$ is such that

$$\alpha = -T'(q).$$

5. *For every $q \in \mathbb{R}$ the following limit exists*

$$h_\mu(f,q) = \lim_{\varepsilon\to 0}\lim_{n\to\infty} -\frac{1}{n(q-1)}\log\int \mu(B_n(x,\varepsilon))^{q-1}\,d\mu, \qquad \text{if } q \neq 1,$$

$$h_\mu(f,q) = \lim_{\varepsilon\to 0}\lim_{n\to\infty} -\frac{1}{n}\int \log\mu(B_n(x,\varepsilon))\,d\mu, \qquad \text{if } q = 1.$$

The quantities $h_\mu(f,q)$ depend continuously on q and

$$h_\mu(f,0) = h_{top}(f), \quad h_\mu(f,1) = h_\mu(f),$$

where $h_{top}(f)$ and $h_\mu(f)$ are the topological and measure-theoretic entropies of f correspondingly. Moreover,

$$h_\mu(f,q) = -\frac{T(q)}{q-1}, \quad q \neq 1.$$

We call $h_\mu(f,q)$ the correlation entropy of order q. The spectrum of correlation entropies is the dynamical analog of the Hentschel–Procaccia and Rényi spectra of generalized dimensions.

5. Final remarks

A. Consider irregular set

$$B = \{x \in X : \ h_\mu(f,x) \text{ does not exist }\}.$$

It was shown in [BS97] that in a number of cases, the set B is either empty or has full topological entropy and Hausdorff dimension.

B. There exists another way of defining local (pointwise) entropies. Namely, consider an arbitrary finite measurable partition ζ of X, we can define a local entropy at x with respect to ξ as follows (if the limit exists)

$$h_\mu(f,x,\zeta) = \lim_{n\to\infty} -\frac{1}{n}\log\mu(\xi^{(n)}(x)),$$

where $\xi^{(n)} = \xi \vee f^{-1}\xi \vee \ldots \vee f^{-n+1}\xi$ and $\xi^{(n)}(x)$ is the element of $\xi^{(n)}$ containing x. We can define a spectrum of local entropies with respect to ξ as follows

$$\mathcal{E}_E(\alpha) = h_{top}(f|_{\{x: h_\mu(f,x,\xi)=\alpha)\}})$$

The situation when ξ is a finite Markov partition has been studied in [BPS97a, BPS97b]. One can eaily check that in this case two spectra coincide.

C. A requirement of the existence of a Markov partition is stronger than a specification property, provided the dynamical system is mixing. Consider a family of one-dimensional maps on the unit interval

$$T_\beta(x) = \beta x \ (\text{mod } 1), \quad \beta > 1.$$

These maps are obviously expansive, moreover, they are expanding. The ergodic properties of T_β depend on the number-theoretic properties of β. In [Sch97] the results on the massiveness of the sets of β's with prescribed dynamical properties were obtained. We cite two of them

i) the set of β's for which T_β has a finite Markov partition is at most countable;

ii) the set of β's for which T_β has the specification property is uncountable and has Hausdorff dimension 1, but still has Lebesgue measure 0.

Therefore, we can see that in the family

$$\{T_\beta |\ \beta > 1\}$$

specification is a much more general property, than the property of having a finite Markov partition.

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FLORIS TAKENS AND EVGENI VERBITSKI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GRONINGEN
P.O.BOX 800, 9700 AV
GRONINGEN
THE NETHERLANDS
E-MAIL: f.takens@math.rug.nl
e.verbitski@math.rug.nl